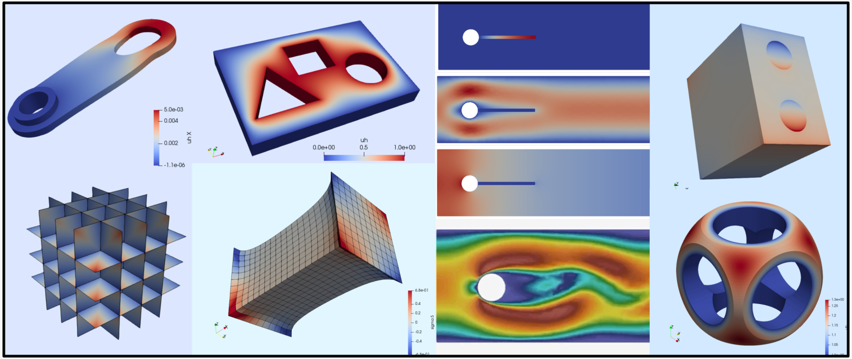
Introduction to the finite element method



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MONASH University

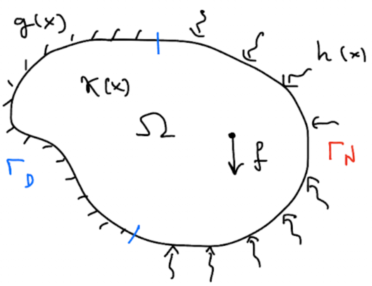
Introduction

Probably, you are familiar with the strong form of PDEs

Example: Poisson equation

$$-\nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma_D \qquad -\kappa \nabla u \cdot n = h \quad \text{on } \Gamma_N$$

- $\Omega \subset \mathbb{R}^D$ is the physical domain,
- Γ_D is the Dirichlet boundary,
- Γ_N is the Neumann boundary



Weak form

PDEs can alternatively be written in weak form

Procedure:

1. Multiply the strong form by a test function v
2. Integrate by parts
3. Apply boundary conditions

$$-\int_\Omega v\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}u) = \int_\Omega \boldsymbol{\nabla}v\cdot(\kappa\boldsymbol{\nabla}u) - \int_{\partial\Omega} v\kappa\boldsymbol{\nabla}u\cdot n = \int_\Omega \boldsymbol{\nabla}v\cdot(\kappa\boldsymbol{\nabla}u) - \int_{\Gamma_N} vh$$

using that $v = 0$ on Γ_D and $\kappa\boldsymbol{\nabla}u\cdot\boldsymbol{n} = h$ on Γ_N

Weak form (II)

Example: Poisson equation

$$\text{Find } u \in V^D \ : \int_\Omega \nabla v \cdot (\kappa \nabla u) = \int_\Omega v f + \int_{\Gamma_N} v h, \quad \forall v \in V^0$$

where V is a function space (crucial for well-posedness) and

$$V^D = \{v \in V : v = g \text{ on } \Gamma_D\}, \qquad V^0 = \{v \in V : v = 0 \text{ on } \Gamma_D\}$$

are the **trial** and **test** spaces, respectively

The weak form is used in finite element methods

Function spaces

The weak form is a variational solution of a quadratic functional

$$u = \arg\min_{u \in V^D} J(u), \qquad J(u) = \int_\Omega \kappa |\nabla u|^2 - \int_\Omega uf - \int_{\Gamma_N} uh$$

It makes sense to consider V as the space in which $J(u) < \infty$ (well-defined)

$$V = H^1(\Omega) = \{u(x) \ : \ \int_\Omega |\nabla u|^2 < \infty\}$$

- V is an infinite-dimensional space of functions
- We need to discretize the problem to obtain a finite-dimensional system of equations (e.g., using polynomials)

Spectral approximation

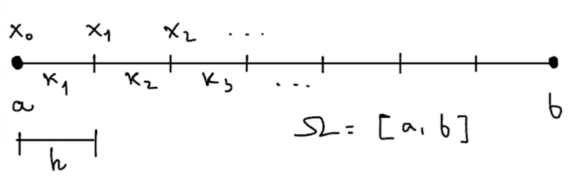
Approximate V by the polynomial space of order p

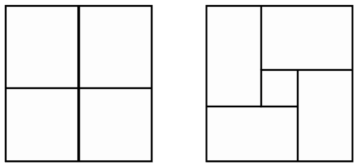
$$\mathcal{P}_p = \{1, x, x^2, \ldots, x^p\}$$

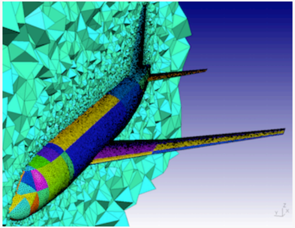
- Hard to deal with geometries that are not boxes
- It exploits the smoothness of the solution (Taylor expansion)

Finite element spaces

Consider a **mesh** $\mathcal{M}_h$, i.e., a partition of Ω into elements / cells (segments, triangles or quadrilaterals, tetrahedra or hexahedra, etc.)



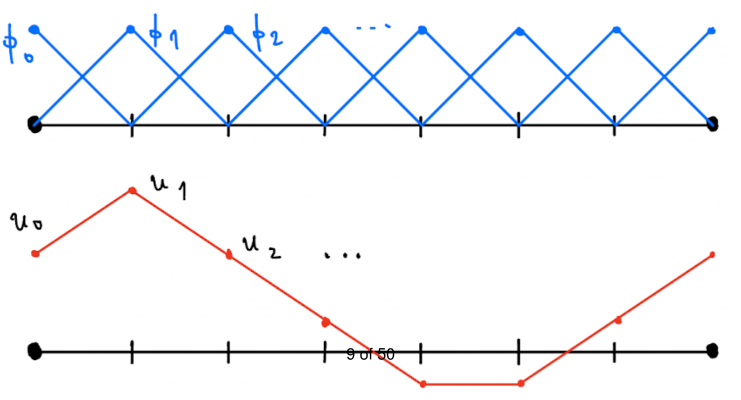




FEM space

A **finite element space** $V_h \subset V$ is a space of piecewise polynomials of order p defined on $\mathcal{M}_h$

$$V_h \doteq \{v_h \in V : v_h|_K \in \mathcal{P}_p, \quad \forall K \in \mathcal{M}_h\}$$



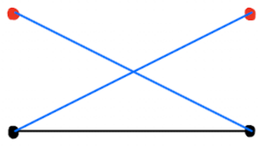
Reference FE

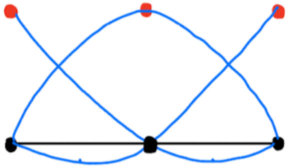
A **reference FE** is composed of:

- A polytope (triangle, square, etc), the reference cell $\hat{K}$
- A reference FE space $\hat{\mathcal{V}}$ of polynomials on $\hat{K}$
- The degrees of freedom (DOFs) that define the *shape functions* basis for $\hat{\mathcal{V}}$

Lagrangian 1D (SEGMENT)

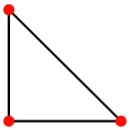
- $\hat{K} = [0, 1]$,
- $\hat{\mathcal{V}} = \{1, x, x^2, \ldots, x^p\} = \mathcal{P}_p$,
- DOFs: Nodal values at $\{0, 1/p, 2/p, \ldots, 1\}$

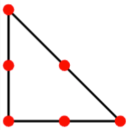




Lagrangian 2D (TRI)

- Triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$
- $\mathcal{P}_p = \{1, x, y, x^2, xy, y^2, \ldots\}$ (tensor product)
- DOFs: Nodal values





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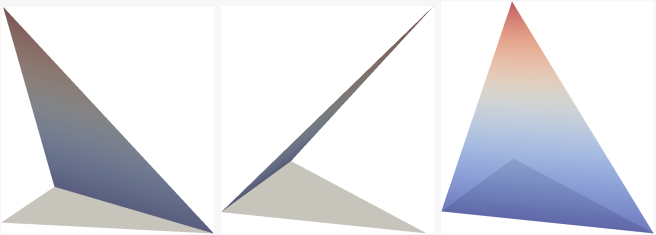

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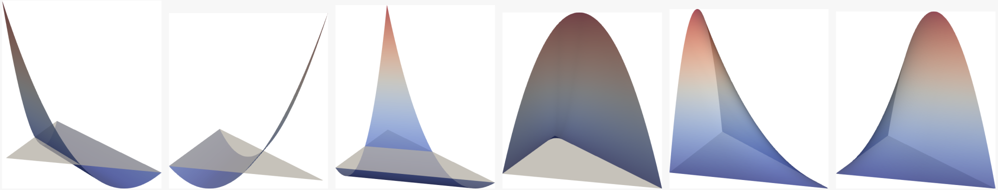
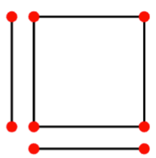
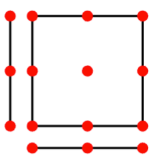


Lagrangian 2D (QUAD)

- $\hat{K} = [0,1]^2$,
- $\mathcal{Q}_p = \{1, x, y, xy, x^2, x^2y, x^2y^2, \ldots\}$
- DOFs: Nodal values





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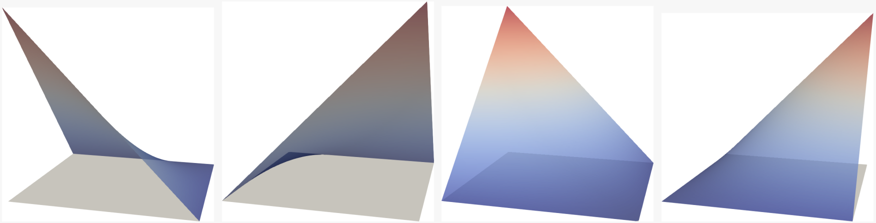

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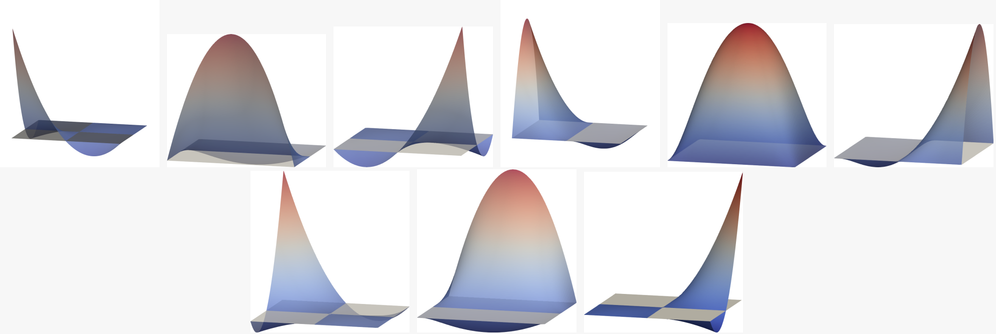
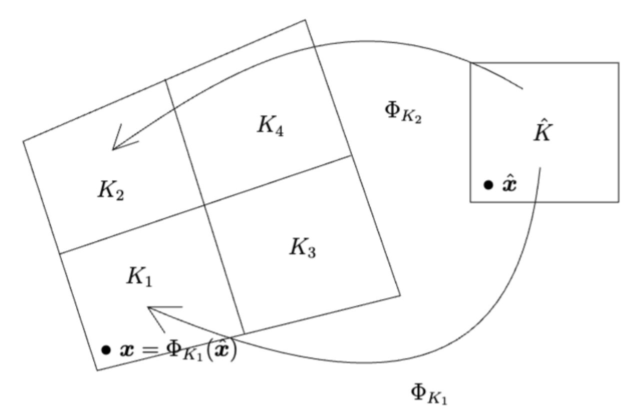


From reference to physical space

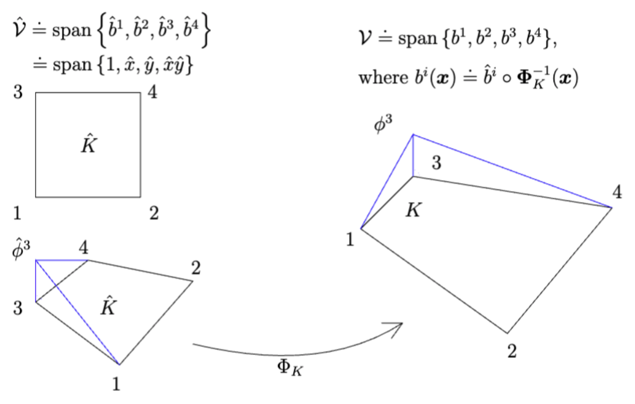
- A **geometric map** $\Phi_K : \hat{K} \to K$



From reference to physical space

- The space to the physical cell K is

$$V_K = \{v \circ \Phi_K^{-1} : v \in \mathcal{P}_p(\hat{K})\}$$



Conformity

Still, there is part of the definition that is not covered:

$$V_h \doteq \{v_h \in V : v_h|_K \in \mathcal{P}_p, \quad \forall K \in \mathcal{M}_h\}$$

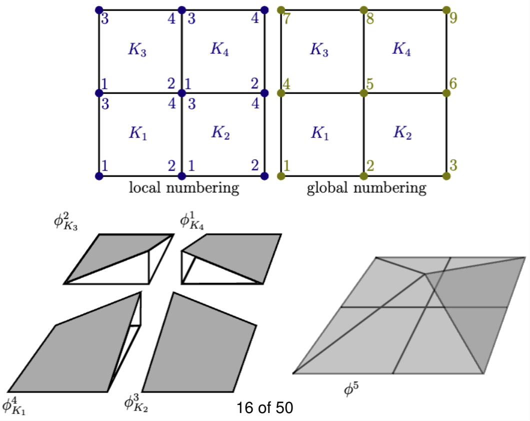
- A discontinuous piecewise polynomial is not in $H^1(\Omega)$
- $\int_\Omega \nabla u_h$ is not bounded
- Piecewise polynomials in $C^0(\Omega)$ are in $H^1(\Omega)$

Enforcing continuity

We must enforce continuity for $V_h \subset H^1(\Omega)$

$$V_h \doteq \{v_h \in C^0(\Omega) : v_h|_K \in \mathcal{P}_p, \quad \forall K \in \mathcal{M}_h\}$$

Using Lagrangian FEs, we just *assemble / glue* together DOFs of adjacent cells



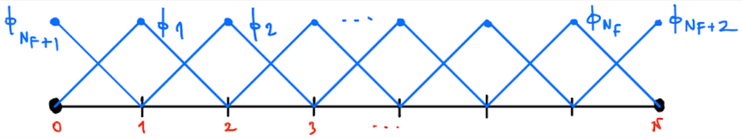
FE Basis

Let us split the Lagrangian nodes $\mathcal{N}$ of the FE space V_h into free nodes $\mathcal{N}_F$ (on $\Omega \subset \Gamma_D$) and Dirichlet nodes $\mathcal{N}_D$ (on Γ_D)

For each node $i \in \mathcal{N}$, we can consider the shape functions (Lagrangian nodes)

$$\phi^i(\boldsymbol{x}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

It returns a basis of $V_h = \{\phi^1, \ldots, \phi^{N_F}, \phi^{N_F+1}, \ldots, \phi^{N_F+N_D}\}$



Galerkin + FEM

FE discretisation of the Poisson equation (using Galerkin method)

$$\text{Find } u_h \in V_h^D \ : \ \int_\Omega \boldsymbol{\nabla} v_h \cdot (\kappa \boldsymbol{\nabla} u_h) = \int_\Omega v_h f + \int_{\Gamma_N} v_h h, \quad \forall v_h \in V_h^0$$

where V_h is a function space (crucial for well-posedness) and

$V_h^D = \{v_h \in V_h : v_h = g$ at nodes on $\Gamma_D\}$, $\qquad V^0 = \{v_h \in V_h : v_h = 0$ at nodes on $\Gamma_D\}$

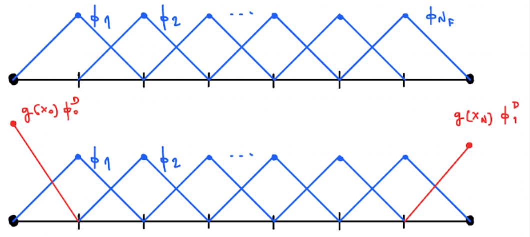
are the **trial** and **test** FE spaces, respectively

Trial/test FE basis

- Test space: A basis for $V_h^0 = \{\phi^1, \dots, \phi^{N_F}\}$
- Trial space: Any function $u_h \in V_h^D$ can be written as

$$u_h = u_h^0 + u_h^D, \qquad u_h^0 = \sum_{i=1}^{N_F} \mathbf{u}^i \phi^i \in V_h^0, \quad u_h^D = \sum_{i=N_F+1}^{N_F+N_D} g(\boldsymbol{x}_i)\phi^i$$

where u_h^D is data (g is given) and $\mathbf{u} \in \mathbb{R}^{N_F}$ is the unknown vector of coefficients



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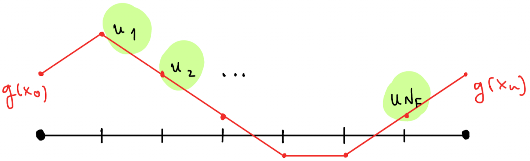

Linear system (I)

- Galerkin formulation:

$$\text{Find } u_h \in V_h^D \;:\; \int_\Omega \boldsymbol{\nabla} v_h \cdot (\kappa \boldsymbol{\nabla} u_h) = \int_\Omega v_h f + \int_{\Gamma_N} v_h h, \quad \forall v_h \in V_h^0$$

- Using the FE basis for the test space

$$\text{Find } u_h \in V_h^D \;:\; \int_\Omega \boldsymbol{\nabla} \phi^i \cdot (\kappa \boldsymbol{\nabla} u_h) = \int_\Omega \phi^i f + \int_{\Gamma_N} \phi^i h, \quad \forall i = 1, \ldots, N_F$$

Linear system (I)

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$$\text{Find } u_h \in V_h^D \;:\; \int_\Omega \boldsymbol{\nabla} v_h \cdot (\kappa \boldsymbol{\nabla} u_h) = \int_\Omega v_h f + \int_{\Gamma_N} v_h h, \quad \forall v_h \in V_h^0$$

- Using the decomposition $u_h = u_h^0 + u_h^D$

$$\text{Find } \mathbf{u} \in \mathbb{R}^{N_F} \;:\; \left[\int_\Omega \boldsymbol{\nabla}\phi^i \cdot (\kappa \boldsymbol{\nabla}\phi^j)\right] \mathbf{u}^j = \int_\Omega \phi^i f + \int_{\Gamma_N} \phi^i h - \int_\Omega \boldsymbol{\nabla}\phi^i \cdot (\kappa \boldsymbol{\nabla} u_h^D),$$

$\forall i = 1, \ldots, N_F$

- We end up with a linear system to be solved $\mathbf{A}\mathbf{u} = \mathbf{b}$

Linear system(II)

- One can compute all the integrals using a **quadrature rule** $\mathcal{Q}$ at the reference element, e.g.,

$$
\begin{aligned}
\int_\Omega \boldsymbol{\nabla}\phi^i \cdot (\kappa \boldsymbol{\nabla}\phi^j) &= \sum_{K\in\mathcal{M}_h} \int_{\hat{K}} J_K^{-T}\hat{\boldsymbol{\nabla}}\hat{\phi}^i \cdot (\kappa J_K^{-T}\hat{\boldsymbol{\nabla}}\hat{\phi}^j) \det(J_K) \\
&= \sum_{K\in\mathcal{M}_h} \sum_{\hat{\boldsymbol{x}}_{\mathrm{gp}\in\mathcal{Q}}} J_K^{-T}\hat{\boldsymbol{\nabla}}\hat{\phi}^i \cdot (\kappa \circ \Phi_K J_K^{-T}\hat{\boldsymbol{\nabla}}\hat{\phi}^j) \det(J_K)|_{\hat{\boldsymbol{x}}_{\mathrm{gp}}} w_{\mathrm{gp}}
\end{aligned}
$$

  where $J_k = \boldsymbol{\nabla}\Phi_K$.
- Usually, we use a Gaussian quadrature $\mathcal{Q}$ that integrates exactly the matrix terms of the linear system
- The **degree** of the quadrature is the maximum order of a polynomial that can be integrated exactly (e.g., $2p$ for FE spaces of order p and a linear geometrical map)

Linear system (III)

- We have started with a PDE in weak form (∞ dimensional space V)
- Using a FE space V_h (finite dimensional polynomial space) we have transformed it into a linear system $\mathbf{A}\mathbf{u} = \mathbf{b}$
- This approximation comes with the price of a **numerical error**

Bounds for numerical errors

Let us define the L^2 and H^1 norms of a function u

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega u^2\right)^{1/2}, \qquad \|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)^{1/2},$$

- The **discretisation error** $e_h = u - u_h$ can be bounded by

$$\|e_h\|_{H^1(\Omega)} \leq Ch^q \|u\|_{H^{q+1}(\Omega)}, \qquad \|e_h\|_{L^2(\Omega)} \leq Ch^{q+1} \|u\|_{H^{q+1}(\Omega)}$$

  for any $q \leq p$, where h is the mesh size and p is the order of the FE space
- The H^{p+1} norm means the L^2-norm of all the derivatives up to $p + 1$ (requires **smoothness**)

Bounds for numerical errors (II)

Assuming the solution is smooth enough ($q = p$),

$$\|e_h\|_{H^1(\Omega)} \leq C_u h^p, \qquad \|e_h\|_{L^2(\Omega)} \leq C_u h^{p+1}$$

Thus,

$$\log \|e_h\|_{H^1(\Omega)} \leq C + p \log h, \qquad \log \|e_h\|_{L^2(\Omega)} \leq C + (p+1) \log h$$

We can check these bounds experimentally in the tutorials

Linear elasticity

Linear elasticity (strong form):

$$\begin{cases} -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \text{ in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma_{\mathrm{D}}, \\ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{h} \text{ on } \Gamma_{\mathrm{N}}. \end{cases}$$

where $\boldsymbol{u}$ is the displacement **vector** and $\boldsymbol{\sigma}(\boldsymbol{u})$ is the stress 2-tensor defined as

$$\boldsymbol{\sigma}(\boldsymbol{u}) \doteq \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\boldsymbol{u}))\, I + 2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}), \qquad \boldsymbol{\varepsilon}(\boldsymbol{u}) \doteq \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right)$$

Weak form (elasticity)

Times $\boldsymbol{v}$ the strong form and integrate by parts:

$$\int_\Omega \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{u})) = \int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\sigma}(\boldsymbol{u}) - \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\sigma}(\boldsymbol{u}) - \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}$$
$$= \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\sigma}(\boldsymbol{u}) - \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{h}$$

Weak form (elasticity)

We get the weak form:

$$\text{Find } \boldsymbol{u} \in \boldsymbol{V}^D \ : \ \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\sigma}(\boldsymbol{u}) = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} + \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{h}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}^0$$

- $\boldsymbol{V} = [H^1(\Omega)]^D$ (Korn's inequality),
- $\boldsymbol{V}^D = \{\boldsymbol{v} \in \boldsymbol{V} \ : \ \boldsymbol{v} = \boldsymbol{g} \text{ on } \Gamma_D\}$ is the trial space
- $\boldsymbol{V}^0 = \{\boldsymbol{v} \in \boldsymbol{V} \ : \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$ is the test space

Finite element space

- We want a FE space $\boldsymbol{V}_h \subset \boldsymbol{V} = [H_0^1(\Omega)]^D$
- Same conformity, i.e., $\boldsymbol{V}_h \subset [C^0(\Omega)]^D$
- $\boldsymbol{V}_h = [V_h]^D$, where V_h is the scalar FE space of the previous section
- All the ideas in the previous section readily apply for each component

Nonlinear problems

Let us consider a nonlinear model problem, p-Laplacian:

$$-\boldsymbol{\nabla} \cdot (|\boldsymbol{\nabla} u|^{p-2} \boldsymbol{\nabla} u) = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

where $p \geq 2$ is a given parameter

The weak form is

$$\text{Find } u \in V^D \; : \int_\Omega |\boldsymbol{\nabla} u|^{p-2} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = \int_\Omega f v, \quad \forall v \in V^0$$

- Same conformity as Poisson, $V^D = H^1(\Omega)$

Newton's method

- Nonlinear problem wrt u
- One can use Newton's method to solve it
- We want to solve $f(u) = 0$ iteratively
- Using the fact that $f(u + \delta u) \approx f(u) + f'(u)\delta u$
- Given u^i

$$f'(u^i)\delta u^{i+1} = -f(u^i), \quad u^{i+1} = u^i + \delta u^{i+1}$$

till convergence

Residual and Jacobian

We better state the problem in terms of the residual:

$$u \in V^D \ : \ r(u,v) = \int_\Omega \nabla v \cdot \left(|\nabla u|^{p-2} \, \nabla u\right) \, \mathrm{d}\Omega - \int_\Omega v \, f \, \mathrm{d}\Omega = 0, \quad \forall v \in V^0$$

We compute the variation of the residual wrt a given direction $\delta u \in V^0$ at $u \in V^D$

$$r(u + \delta u, v) \approx r(u,v) + \frac{\partial r(u,v)}{\partial u} \delta u$$

where $j(\partial u, u, v) = \frac{\partial r(u,v)}{\partial u}$ is the **Jacobian** evaluated at $u \in U_g$, which is the bilinear form

$$[j(u,v)\delta u = \int_\Omega \nabla v \cdot \left(|\nabla u|^{p-2} \, \nabla \delta u\right) \, \mathrm{d}\Omega + (p-2) \int_\Omega \nabla v \cdot \left(|\nabla u|^{p-4} (\nabla u \cdot \nabla \delta u) \nabla u\right) \, \mathrm{d}\Omega.$$

Discrete problem

Using Newton + FEM:

$$\text{Find } \delta u_h^{i+1} \in V_h^0 \; : \; j(\delta u_h^{i+1}, u_h^i, v_h) = -r(u_h^i, v_h), \quad \forall v_h \in V_h^0$$

- $j(\delta u_h^{i+1}, u_h^i, v_h)$ is a linear system to be solved at each nonlinear iteration
- After linearisation, we can apply the same ideas as in the previous section
- We can compute the expression of the Jacobian by hand or using automatic differentiation

Heat equation in weak form:

$$\text{Find } u \in V^D \ : \int_\Omega v \partial_t u + \int_\Omega \nabla v \cdot (\kappa \nabla u) = \int_\Omega v f + \int_{\Gamma_N} v h, \quad \forall v \in V^0$$

Semi-discretised problem (using ideas above, only discretise in space):

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} = \mathbf{b}, \qquad \dot{\mathbf{u}} = -\mathbf{M}^{-1}\mathbf{A}\mathbf{u} + \mathbf{M}^{-1}\mathbf{b}$$

- $\mathbf{M}_{ij} = \int \phi^i(\boldsymbol{x})\phi^j(\boldsymbol{x})$ is the mass matrix
- $\mathbf{A}_{ij} = \int \kappa(\boldsymbol{x})\nabla\phi^i(\boldsymbol{x}) \cdot \nabla\phi^j(\boldsymbol{x})$ is the stiffness matrix
- $\mathbf{b}$ is the load vector
- $\mathbf{u}$ is the vector of unknowns

Time discretisation: Create a 1D partition of the time interval $[0, T]$, $\mathcal{T}_h = \{0 = t_0 < t_1 < \ldots < t_N = T\}$, with $\Delta t = t_{n+1} - t_n$ and $t_n = n\Delta t$.

- Backward Euler (Implicit, 1st order)

$$\mathbf{M}\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}\mathbf{u}^{n+1} = \mathbf{b}, \qquad (\mathbf{M} + \Delta t\mathbf{A})\,\mathbf{u}^{n+1} = \Delta t\mathbf{b} + \mathbf{M}\mathbf{u}^n$$

- Crank-Nicolson (Implicit, 2nd order)

$$\mathbf{M}\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}\mathbf{u}^{n+1/2} = \mathbf{b}, \qquad (\mathbf{M} + \Delta t/2\mathbf{A})\,\mathbf{u}^{n+1/2} = \Delta t/2\mathbf{b} + \mathbf{M}\mathbf{u}^n$$

  where $\mathbf{u}^{n+1/2} = 1/2(\mathbf{u}^{n+1} + \mathbf{u}^n)$

- Forward Euler (Explicit, 1st order, conditionally stable, $\Delta t < Ch^2$)

$$\mathbf{M}\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}\mathbf{u}^n = \mathbf{b}, \qquad \mathbf{M}\mathbf{u}^{n+1} = \Delta t\mathbf{b} + \mathbf{M}\mathbf{u}^n - A\mathbf{u}^n$$

- Runge-Kutta methods (implicit, explicit, IMEX), ...

Computational cost

- Solve a linear system at each time step
- Implicit methods, system matrix $\mathbf{M} + c\Delta t\mathbf{A}$
- Explicit methods, system matrix $\mathbf{M}+$ (much cheaper, better conditioned, but stringent condition for stability)

Stokes problem

Strong form: Find $\boldsymbol{u} \in \boldsymbol{V}^D$ and $p \in Q$ such that

$$\begin{cases} -\boldsymbol{\nabla} \cdot \mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma_D, \\ \mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n} - p\boldsymbol{n} = \boldsymbol{h} & \text{on } \Gamma_N. \end{cases}$$

Stokes problem

Testing with $\boldsymbol{v} \in \boldsymbol{V}^0$ and integrating by parts

$$-\int_\Omega \boldsymbol{v}\boldsymbol{\nabla} \cdot \mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = \int_\Omega \boldsymbol{\nabla}\boldsymbol{v} : \mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) - \int_{\Gamma_N} \boldsymbol{v} \cdot \mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n}$$
$$= \int_\Omega \mu\,\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{u}) - \int_{\Gamma_N} \boldsymbol{v} \cdot \mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{n}$$

In order to have the right stresses on the Neumann boundary, we integrate by parts the pressure term

$$\int_\Omega \boldsymbol{\nabla} p \cdot \boldsymbol{v} = -\int_\Omega p\boldsymbol{\nabla} \cdot \boldsymbol{v} + \int_{\Gamma_N} p\boldsymbol{v} \cdot \boldsymbol{n}$$

Weak form

Adding together with mass conservation, we get the weak form: find $\boldsymbol{u} \in \boldsymbol{V}^D$ and $p \in Q$ such that

$$\int_\Omega \mu \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{u}) - \int_\Omega p \boldsymbol{\nabla} \cdot \boldsymbol{v} + \int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{u} = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} + \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{h}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}^0,\ \forall q \in Q$$

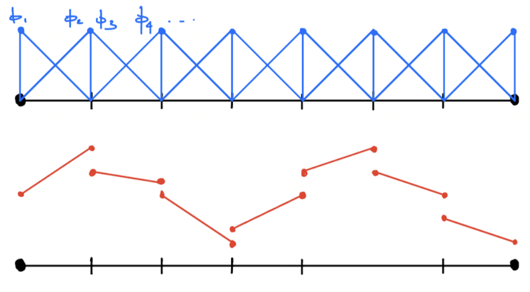
- $\boldsymbol{V} = [H^1(\Omega)]^D$ (Korn's inequality),
- $\boldsymbol{V}^D = \{\boldsymbol{v} \in \boldsymbol{V} \ : \ \boldsymbol{v} = \boldsymbol{g} \text{ on } \Gamma_D\}$ is the trial space
- $\boldsymbol{V}^0 = \{\boldsymbol{v} \in \boldsymbol{V} \ : \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$ is the test space
- $Q = L^2(\Omega)$ (no derivatives, no continuity required in FEM)

L^2-conformity

- We need to define a FE space $Q_h \subset Q = L^2(\Omega)$

$$Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_p(K) \text{ or } \mathcal{Q}_p(K)\ \forall K \in \mathcal{M}_h\}$$

- No inter-element continuity required by $L^2(\Omega)$
- Simplified version of the previous section (not gluing required)



Mixed FEM

- We can use discontinuous FE spaces
- We can use continuous FE spaces too
- However, we need to satisfy the so-called inf-sup stability condition

Mixed FEM (II)

Suitable spaces for the Stokes problem:

- Tris/Tets: $\mathcal{P}_k \times \mathcal{P}_{k-1}$ Taylor-Hood element, $k \geq 2$
- Quads/Hexs: $\mathcal{Q}_k \times \mathcal{Q}_{k-1}$ Taylor-Hood element, $k \geq 2$
- Quads/Hexs: $\mathcal{Q}_{k+1} \times \mathcal{P}_k^-$, $k \geq 2$

Note: $\mathcal{P}_k^-$ means discontinuous polynomials of degree k (analogously for $\mathcal{Q}_k^-$)

Darcy equation

Strong form: Find $\boldsymbol{u} \in \boldsymbol{V}^D$ and $p \in Q$ such that

$$\begin{cases} \boldsymbol{u} + \kappa \boldsymbol{\nabla} p = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} = f & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma_D, \\ p = g & \text{on } \Gamma_N. \end{cases}$$

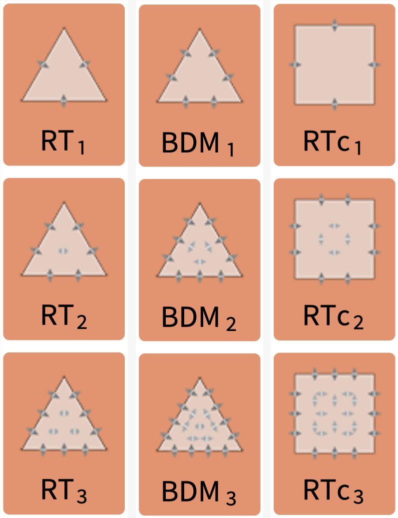
Weak form

Weak form: find $(u, p) \in H(\mathrm{div}; \Omega) \times L^2(\Omega)$ such that $a((u, p), (v, q)) = b(v, q)$ for all $(v, q) \in H(\mathrm{div}; \Omega) \times L^2(\Omega)$, where

$$
\begin{aligned}
a((u,p),(v,q)) &\doteq \int_\Omega v \cdot u \, \mathrm{d}\Omega - \int_\Omega (\nabla \cdot v)\, p \, \mathrm{d}\Omega + \int_\Omega q \, (\nabla \cdot u) \, \mathrm{d}\Omega, \\
b(v,q) &\doteq \int_\Omega q \, f \, \mathrm{d}\Omega - \int_{\Gamma_\mathrm{N}} (v \cdot n) \, g \, \mathrm{d}\Gamma.
\end{aligned}
$$

- Only control on $\nabla \cdot \boldsymbol{u}$ needed
- It only implies continuity of $\boldsymbol{u} \cdot \boldsymbol{n}$ on element boundaries

Div-conforming FEM



- $\mathcal{RT}_k = [\mathcal{P}_k]^D + \boldsymbol{x}\mathcal{P}_k$
- $\mathcal{BDM}_k = [\mathcal{P}_k]^D$
- $\mathcal{RT}_k = \mathcal{Q}_{k+1,k} \times \mathcal{Q}_{k,k+1}$
- DOFs are normal fluxes on the faces of the elements (for div-conformity) + ...

Stable pairs

Stable flux / pressure spaces:

- Tris/tets: $\mathcal{RT}_k \times \mathcal{P}_k^-$
- Tri/tets: $\mathcal{BDM}_{k+1} \times \mathcal{P}_k^-$
- Quads/hex: $\mathcal{RT}_k \times \mathcal{Q}_k^-$

Inf-sup condition

There exists $\beta > 0$ such that

$$\inf_{q \in Q_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h^0} \frac{\int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|_{H^1(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta$$

Analogously for $\mathbf{curl}$ operator (Maxwell equations, not covered in this tutorial)